

Interior perfect fluid scalar-tensor solution

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Abstract

We present a new exact perfect fluid interior solution for a particular scalar-tensor theory. The solution is regular everywhere and has a well defined boundary where the fluid pressure vanishes. The metric and the dilaton field match continuously the external solution.

Exact solutions provide a route to better and more deep understanding of the inherent nonlinear character of gravity. One of the most difficult tasks is the construction of interior perfect fluid solutions which are of great astrophysical interest. There are some known exact perfect fluid interior solutions in general relativity [1]. Most of them, however, are not physically acceptable: the perfect fluid satisfies unrealistic equations of state, the natural energy conditions are violated, the space-time is singular or the solutions have not a well defined boundary. Nevertheless, those solutions are useful since they provide some non-perturbative insight into the highly nonlinear gravitational phenomena. On the other hand, the exact solutions, even unrealistic, could serve as tests for checking the computer codes which is important for the advent of numerical relativity.

The most realistic exact interior general relativistic solution is the Schwarzschild interior solution describing a static, spherically symmetric perfect fluid star with a uniform density. This solution has well defined boundary where the fluid pressure vanishes and matches continuously the external Schwarzschild solution on that boundary. The interior Schwarzschild solution qualitatively describes the general case of a static, spherically symmetric perfect fluid star in general relativity and in particular, it predicts the existence of an upper limit for the ratio of the star mass to the star radius, known in the general case as the Buchdahl inequality [2].

General relativity is not the only viable theory of gravity. The most natural alternatives of general relativity are the scalar-tensor theories of gravity [3],[4]. In these theories gravity is mediated not only by the metric of space-time but also by a scalar field. The importance of scalar-tensor theories is related to the unifying theories, such as string theory and higher dimensions gravity theories, which in their low energy limit predict the existence of a scalar partner of the tensor graviton -the dilaton.

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Scalar-tensor theories contain arbitrary functions of the dilaton field determining the coupling between the dilaton and the matter fields. With a proper choice of the coupling functions, scalar-tensor theories can pass all experimental constraints. In the weak field regime the predictions of the scalar-tensor theories are very close to those of general relativity. However, in the strong field regime the scalar-tensor theories can behave in a drastically different way. Such a strong non-perturbative effect, called spontaneous scalarization, was discovered numerically for neutron stars a decade ago [5]. That is why it would be nice and important if we can have exact interior solutions giving even only a qualitative picture of the nonlinear structure of a static, spherically symmetric star in the presence of the dilaton field. Solving the interior problem in scalar-tensor theories, however, is much difficult and seems to be a hopeless task in the general case due to the presence of the arbitrary coupling function $\alpha(\varphi)$ in the field equations (see eqs. (6) below). Forced by the mathematical difficulties, we shall focus our attention on a particular scalar-tensor theory which allow us to find a sufficiently realistic interior solution with the hope that it would give qualitative insight into the general case of a static, spherically symmetric star. Let us mention that some exact "interior" solutions were previously found in Brans-Dicke theory for perfect fluid with equation of state $p = \gamma\rho$ [6],[7],[8]. Those solutions, however, are singular at the center and have no well defined boundary where the pressure vanishes.

The general form of the extended gravitational action in scalar-tensor theories is

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-\tilde{g}} \left(F(\Phi) \tilde{R} - Z(\Phi) \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2U(\Phi) \right) + S_m[\Psi_m; \tilde{g}_{\mu\nu}]. \quad (1)$$

Here, G_* is the bare gravitational constant, \tilde{R} is the Ricci scalar curvature with respect to the space-time metric $\tilde{g}_{\mu\nu}$. The dynamics of the scalar field Φ depends on the functions $F(\Phi)$, $Z(\Phi)$ and $U(\Phi)$. In order for the gravitons to carry positive energy the function $F(\Phi)$ must be positive. The nonnegativity of the dilaton field energy requires that $2F(\Phi)Z(\Phi) + 3[dF(\Phi)/d\Phi]^2 \geq 0$. The action of matter depends on the material fields Ψ_m and the space-time metric $\tilde{g}_{\mu\nu}$. It should be noted that the stringy generated scalar-tensor theories, in general, admit a direct interaction between the matter fields and the dilaton in the Jordan (string) frame. Here we consider the phenomenological case when the matter action does not involve the dilaton field in order for the weak equivalence principle to be satisfied.

It is much more convenient from a mathematical point of view to analyze the scalar-tensor theories with respect to the conformally related Einstein frame given by the metric:

$$g_{\mu\nu} = F(\Phi) \tilde{g}_{\mu\nu}. \quad (2)$$

Further, let us introduce the scalar field φ (the so called dilaton) via the equation

$$\left(\frac{d\varphi}{d\Phi} \right)^2 = \frac{3}{4} \left(\frac{d \ln(F(\Phi))}{d\Phi} \right)^2 + \frac{Z(\Phi)}{2F(\Phi)} \quad (3)$$

and define

$$\mathcal{A}(\varphi) = F^{-1/2}(\Phi) \ , 2V(\varphi) = U(\Phi)F^{-2}(\Phi). \quad (4)$$

In the Einstein frame the action (1) takes the form

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} (R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 4V(\varphi)) \\ + S_m[\Psi_m; \mathcal{A}^2(\varphi)g_{\mu\nu}] \quad (5)$$

where R is the Ricci scalar curvature with respect to the Einstein metric $g_{\mu\nu}$. Then, the Einstein frame field equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_* T_{\mu\nu} + 2\partial_\mu \varphi \partial_\nu \varphi \\ - g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - 2V(\varphi)g_{\mu\nu} \ , \\ \nabla^\mu \nabla_\mu \varphi = -4\pi G_* \alpha(\varphi)T + \frac{dV(\varphi)}{d\varphi} \ , \quad (6)$$

$$\nabla_\mu T_\nu^\mu = \alpha(\varphi)T \partial_\nu \varphi \ .$$

Here $\alpha(\varphi) = d \ln(\mathcal{A}(\varphi))/d\varphi$, and the Einstein frame energy-momentum tensor $T_{\mu\nu}$ is related to the Jordan frame one $\tilde{T}_{\mu\nu}$ via $T_{\mu\nu} = \mathcal{A}^2(\varphi)\tilde{T}_{\mu\nu}$. In the case of a perfect fluid one has

$$\begin{aligned} \rho &= \mathcal{A}^4(\varphi)\tilde{\rho}, \\ p &= \mathcal{A}^4(\varphi)\tilde{p}, \\ u_\mu &= \mathcal{A}^{-1}(\varphi)\tilde{u}_\mu. \end{aligned} \quad (7)$$

In what follows we will consider the case $V(\varphi) = 0$.

We have found an exact solution for the following scalar-tensor theory:

$$\mathcal{A}^2(\varphi) = e^{-\frac{2b\varphi}{(2-a)}} \left[(3-a)e^{\frac{\varphi}{b}} - (2-a) \right]^{\frac{2b^2}{(2-a)(3-a)}} \quad (8)$$

giving the coupling function

$$\alpha(\varphi) = b \frac{1 - e^{\frac{\varphi}{b}}}{(3-a)e^{\frac{\varphi}{b}} - (2-a)} \quad (9)$$

where $a^2 + b^2 = 1$.

In addition to the parameters a and b coming from the particular scalar-tensor theory, the exact solution depends also on two parameters μ and $R > 0$. In order for our solution to have physical meaning (i.e. positive mass, positive fluid energy density and pressure) the parameters μ and a must satisfy $\mu a > 0$.

The space-time metric is given by

$$ds^2 = -e^{2a\lambda} dt^2 + e^{2(1-a)\lambda} \frac{dr^2}{1 - \frac{2\mu G_* r^2}{R^3}} + e^{2(1-a)\lambda} r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (10)$$

where

$$e^{2\lambda(r)} = \frac{1}{4} \left[3 \left(1 - \frac{2\mu G_*}{R} \right)^{1/2} - \left(1 - \frac{2\mu G_* r^2}{R^3} \right)^{1/2} \right]^2. \quad (11)$$

The dilaton field, the pressure, the fluid energy density and the fluid four-velocity are respectively given by :

$$\varphi(r) = b\lambda(r) - b \ln \left[\frac{3}{2} \left(1 - \frac{2\mu G_*}{R} \right)^{1/2} \right], \quad (12)$$

$$p(r) = \frac{\mu}{\frac{4\pi}{3} R^3} \frac{\left(1 - \frac{2\mu G_* r^2}{R^3} \right)^{1/2} - \left(1 - \frac{2\mu G_*}{R} \right)^{1/2}}{3 \left(1 - \frac{2\mu G_*}{R} \right)^{1/2} - \left(1 - \frac{2\mu G_* r^2}{R^3} \right)^{1/2}} e^{-2(1-a)\lambda(r)}, \quad (13)$$

$$\rho(r) = \frac{\mu a}{\frac{4\pi}{3} R^3} e^{-2(1-a)\lambda(r)} - 3(1-a)p(r), \quad (14)$$

$$u = e^{-a\lambda(r)} \frac{\partial}{\partial t}. \quad (15)$$

The solution has well defined boundary $r = R$ where, as it can be seen, the pressure vanishes, $p(R) = 0$, which determines the star surface. In order for the solution to be physically regular everywhere for $0 \leq r \leq R$, i.e.

$$0 < \mathcal{A}^2[\varphi(r)] < \infty, \quad 0 \leq \rho(r) < \infty, \quad 0 \leq p(r) < \infty, \quad 0 < e^{\lambda(r)} < \infty, \quad (16)$$

the parameters μ and R must satisfy the inequality

$$\frac{2|\mu|G_*}{R} < \frac{2}{3}|a| \left(1 - \frac{a}{6} \right). \quad (17)$$

In contrast to the pressure, the fluid energy density does not vanish on the star surface:

$$\rho(R) = \frac{\mu a}{\frac{4\pi}{3} R^3} e^{-2(1-a)\lambda(R)}. \quad (18)$$

The metric and the dilaton field of the interior solution match continuously the external solution

$$ds^2 = - \left(1 - \frac{2\mu G_*}{r}\right)^a dt^2 + \left(1 - \frac{2\mu G_*}{r}\right)^{-a} dr^2 + \left(1 - \frac{2\mu G_*}{r}\right)^{1-a} r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (19)$$

$$\varphi(r) = \frac{b}{2} \ln \left(1 - \frac{2\mu G_*}{r}\right) - b \ln \left[\frac{3}{2} \left(1 - \frac{2\mu G_*}{R}\right)^{1/2} \right]. \quad (20)$$

The tensor mass (i.e. the ADM mass in the Einstein frame) can be easily calculated using (19) and is given by $M_T = \mu a$. The inequality (17) then may be written in the form

$$\frac{2M_T G_*}{R} < \frac{2}{3} |a|^2 \left(1 - \frac{a}{6}\right) \quad (21)$$

which can be considered as a (Einstein frame) scalar-tensor version of the Buchdahl inequality.

The asymptotic value of the dilaton field at spatial infinity is

$$\varphi_\infty = \lim_{r \rightarrow \infty} \varphi(r) = -b \ln \left[\frac{3}{2} \left(1 - \frac{2\mu G_*}{R}\right)^{1/2} \right] \quad (22)$$

and, obviously, is different from zero. It should be noted that there is a solution with zero asymptotic value of the dilaton. Indeed, it is not difficult to see that the field equations (6) for the particular scalar-tensor theory defined by (8) and (9) have a solution with a trivial dilaton field, $\varphi = 0$ (i.e. pure general relativistic solution).

The Jordan frame solution can be obtained using eqs. (2), (4) and (7). Since the expressions of the metric, fluid energy density and pressure are rather involved we will not present them in explicit form. However, the qualitative behavior of the fluid pressure, energy density and the gravitational scalar inside the star in the physical Jordan frame can be seen on the figures.

The exact solution presented in this work seems to be the first regular interior solution in scalar-tensor theories of gravity.

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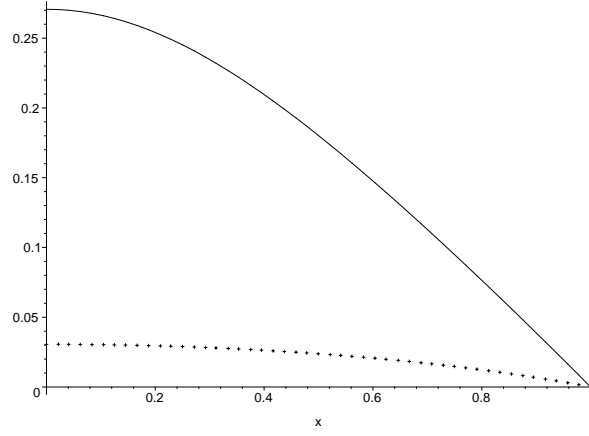


Figure 1: Jordan frame pressure \tilde{p} (in units $\frac{|\mu|}{\frac{4\pi}{3}R^3}$) versus the radial coordinate $x = \frac{r}{R}$ for $|a| = 0.9$ and $2|\mu|G_*/R = 0.2$. The solid line represents the case with $a > 0$.

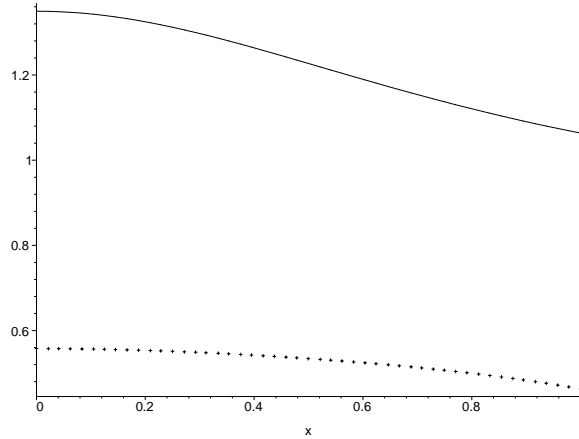


Figure 2: Jordan frame energy density $\tilde{\rho}$ (in units $\frac{|\mu|}{\frac{4\pi}{3}R^3}$) versus the radial coordinate $x = \frac{r}{R}$ for $|a| = 0.9$ and $2|\mu|G_*/R = 0.2$. The solid line represents the case with $a > 0$.

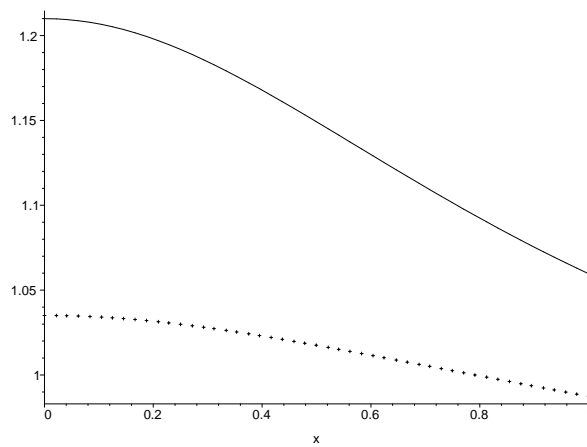


Figure 3: The gravitational scalar $F(\Phi)$ as a function of $x = \frac{r}{R}$ for $|a| = 0.9$ and $2|\mu|G_*/R = 0.2$. The solid line represents the case with $a > 0$.